AN5023

Sensor Fusion Kalman Filters

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Document information

Info	Content
Abstract	This application note documents the mathematics used by the two Kalman filters which implement the fusion of i) accelerometer, magnetometer and gyroscope data and ii) accelerometer and gyroscope data.

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1. Introduction

1.1 Summary

This application note documents the mathematics used by the two Kalman filters which implement the fusion of i) accelerometer, magnetometer and gyroscope data and ii) accelerometer and gyroscope data. These two algorithms are labeled "Gaming Handset" and "Gyro Stabilized Compass" in the NXP Sensor Fusion Toolbox for Windows.

The NXP Application Note (*AN5018) Basic Kalman Filter Theory* provides an introduction to the mathematics of Kalman filters. The sensor fusion Kalman filters are a variant termed i) 'complementary', in that orientation estimates are provided independently by the gyroscope and by the combination of accelerometer and magnetometer and ii) 'indirect' in that the Kalman filter tracks the process error rather than the underlying process itself.

Section 2 describes the estimation of the levels of sensor noise, acceleration and magnetic interference impacting i) the accelerometer estimate of the gravity vector and ii) the magnetometer estimate of the geomagnetic vector.

Section 3 derives expressions for the quaternion orientation errors i) between the gyroscope and accelerometer estimates of the gravity vector and ii) between the gyroscope and magnetometer estimates of the geomagnetic vector.

Section 4 documents the model for the gyroscope sensor's zero rate offset error which is the primary source of error in the gyroscope estimate of orientation. Unlike the accelerometer, the gyroscope is essentially insensitive to acceleration and, unlike the magnetometer, is completely insensitive to magnetic fields.

Section 5 derives the Kalman filter which corrects the gravity and geomagnetic error quaternions derived in section 3 using the noise estimates from section 2 to weight the relative confidence in the gyroscope, accelerometer and magnetometer measurements.

Section 6 derives a simplified variant of the Kalman filter of section 5 designed for use in systems with accelerometer and gyroscope sensors only. Here only the gravity error quaternion is updated according to the relative confidence in the accelerometer and gyroscope measurements.

1.2 Terminology

Symbol	**Definition**
Right superscript -	Denotes an *a priori* estimate made before correction by the Kalman filter.
Right superscript +	Denotes an *a posteriori* estimate made after application of the Kalman filter.
Subscript ε	Denotes an error component.
Subscript k	Refers to sampling iteration k.
Right subscript G	Denotes an accelerometer measurement or estimate.
Right subscript M	Denotes a magnetometer measurement or estimate.
Right subscript Y	Denotes a gyroscope measurement or estimate.

Symbol	Definition
Left superscript G	Denotes that the measurement or estimate is in the global (earth) frame.
Left superscript S	Denotes that the measurement or estimate is in the sensor frame.
${}^{G}\boldsymbol{a}_k, {}^{S}\boldsymbol{a}_k$	Acceleration in the global and sensor frames (units *g*) at iteration k
$\boldsymbol{A}_k$	Linear prediction or state matrix for the error process $\boldsymbol{x}_{\varepsilon,k}$ at iteration k
$\boldsymbol{b}_k$	Gyroscope zero rate offset vector (units deg/s) at iteration k
$\boldsymbol{b}_{\varepsilon,k}$	Gyroscope zero rate offset error vector (units deg/s) at iteration k
B	The local geomagnetic field strength (units µT)
${}^{S}\boldsymbol{B}_{c,k}$	Calibrated magnetometer measurement in the sensor frame (units µT) at iteration k
$\boldsymbol{C}_k$	The measurement matrix relating the measured error process $\boldsymbol{z}_{\varepsilon,k}$ to the underlying error process $\boldsymbol{x}_{\varepsilon,k}$ at iteration k $\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{C}_k\boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k$
${}^{G}\boldsymbol{d}_k, {}^{S}\boldsymbol{d}_k$	The magnetic disturbance in global and sensor frames (units µT) at iteration k
$E[]$	Expectation operator
g	Magnitude of the gravity vector with magnitude 1 *g* in all measurement frames
${}^{G}\boldsymbol{g}_k, {}^{S}\boldsymbol{g}_k$	The gravity vector in the global and sensor frames (units *g*) at iteration k
${}^{S}\boldsymbol{G}_{c,k}$	Calibrated accelerometer measurement in the sensor frame (units *g*) at iteration k
$\boldsymbol{I}$	3 by 3 identity matrix
$\boldsymbol{I}_n$	n by n identity matrix
$\boldsymbol{K}_k$	The Kalman filter gain matrix at iteration k
${}^{G}\boldsymbol{m}_k, {}^{S}\boldsymbol{m}_k$	The geomagnetic vector in the global and sensor frames (units µT) at iteration k
$\hat{\boldsymbol{n}}$	Normalized rotation axis
$\boldsymbol{0}_n$	n by n zero matrix
$\boldsymbol{P}_k^-$	*A priori* covariance matrix.
$\boldsymbol{P}_k^+$	*A posteriori* covariance matrix
$q = \{q_0, q_1, q_2, q_3\}$	Orientation quaternion transforming from global to sensor frame as a result of coordinate system rotation
q_k	Orientation quaternion transforming from global to sensor frame at iteration k
$\hat{q}_k^-$	*A priori* estimate of the orientation quaternion q_k at iteration k
$\hat{q}_k^+$	*A posteriori* estimate of the orientation quaternion q_k at iteration k
$q_{zg\varepsilon,k}, \boldsymbol{q}_{zg\varepsilon,k}$	Measurement rotation quaternion relating the error between the accelerometer and the *a priori* (gyroscope) estimates of the gravity vector. $q_{zg\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{zg\varepsilon,k}$ is the vector component.

Symbol	Definition
$q_{zm\varepsilon,k}, \boldsymbol{q}_{zm\varepsilon,k}$	Measurement rotation quaternion relating the error between the magnetometer and the *a priori* (gyroscope) estimates of the geomagnetic vector. $q_{zm\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{zm\varepsilon,k}$ is the vector component.
$q_{g\varepsilon,k}, \boldsymbol{q}_{g\varepsilon,k}$	The orientation tilt error quaternion relative to the true gravity vector modeled in the Kalman filter. $q_{g\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{g\varepsilon,k}$ is the vector component.
$q_{m\varepsilon,k}, \boldsymbol{q}_{m\varepsilon,k}$	The orientation tilt error quaternion relative to the true geomagnetic vector modeled in the Kalman filter. $q_{m\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{m\varepsilon,k}$ is the vector component.
$Q_{a,k}$	Acceleration variance (units *g*^{2}) at iteration k: $Q_{a,k} = E[\|{}^S\boldsymbol{a}_k\|^2]$
$Q_{d,k}$	Magnetic disturbance variance (units µT^2) at iteration k $Q_{d,k} = E[\|{}^S\boldsymbol{d}_k\|^2]$
$Q_{vB,k}$	Magnetometer sensor noise variance (units µT^2) at iteration k $Q_{vB,k} = E\left[\left\|\boldsymbol{v}_{B,k}\right\|^2\right]$
$Q_{vG,k}$	Accelerometer sensor noise variance (units *g*^{2}) at iteration k $Q_{vG,k} = E\left[\left\|\boldsymbol{v}_{G,k}\right\|^2\right]$
$Q_{vY,k}$	Gyroscope sensor noise variance (units deg^2/s^2) at iteration k $Q_{vG,k} = E\left[\left\|\boldsymbol{v}_{Y,k}\right\|^2\right]$
$Q_{wb,k}$	Covariance of gyroscope zero rate offset random walk (units deg^2/sec^2) at iteration k $Q_{wb,k} = E\left[\left\|\boldsymbol{w}_{b,k}\right\|^2\right]$
$\boldsymbol{Q}_{w,k}$	Covariance matrix of noise process $\boldsymbol{w}_k$ in the underlying process $\boldsymbol{x}_{\varepsilon,k}$ $\boldsymbol{Q}_{w,k} = cov\{\boldsymbol{w}_k, \boldsymbol{w}_k\} = E[\boldsymbol{w}_k \boldsymbol{w}_k{}^T]$
$\boldsymbol{Q}_{v,k}$	Covariance matrix of measurement noise process $\boldsymbol{v}_k$ in the measured process $\boldsymbol{z}$ $\boldsymbol{Q}_{v,k} = cov\{\boldsymbol{v}_k, \boldsymbol{v}_k\} = E[\boldsymbol{v}_k \boldsymbol{v}_k{}^T]$
$\boldsymbol{R}_k$	Rotation or orientation matrix transforming from global to sensor frame at iteration k
$\hat{\boldsymbol{R}}_k^-$	*A priori* estimate of the orientation matrix $\boldsymbol{R}_k$ at iteration k
$\hat{\boldsymbol{R}}_k^+$	*A posteriori* estimate of the orientation matrix $\boldsymbol{R}_k$ at iteration k
$\boldsymbol{r}, \boldsymbol{s}$	Arbitrary vectors
$\boldsymbol{v}_k$	Additive noise in the Kalman filter measurement error process $\boldsymbol{z}_{\varepsilon,k}$: $\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k$
$\boldsymbol{v}_{B,k}$	Magnetometer sensor additive noise (units µT) at iteration k
$\boldsymbol{v}_{G,k}$	Accelerometer sensor additive noise (units *g*) at iteration k
$\boldsymbol{v}_{Y,k}$	Gyroscope sensor additive noise (units deg/s) at iteration k
$\boldsymbol{v}_{qzg,k}$	Noise in the measured gravity tilt error quaternion $\boldsymbol{q}_{zg\varepsilon,k}$ at iteration k
$\boldsymbol{v}_{qzm,k}$	Noise in the measured geomagnetic tilt error quaternion $\boldsymbol{q}_{zm\varepsilon,k}$ at iteration k
$\boldsymbol{w}_k$	Additive noise in error of underlying Kalman filter process $\boldsymbol{x}_{\varepsilon,k}$ $\boldsymbol{x}_{\varepsilon,k} = \boldsymbol{A}_k \boldsymbol{x}_{\varepsilon,k-1} + \boldsymbol{w}_k$

Symbol	Definition
$\boldsymbol{w}_{b,k}$	Driving noise for gyroscope offset random walk (units deg/s) at iteration k
$\boldsymbol{x}_{\varepsilon,k}$	The Kalman filter error process at iteration k
$\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$	The *a priori* estimate of the Kalman filter error process $\boldsymbol{x}_{\varepsilon,k}$
$\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$	The *a posteriori* estimate of the Kalman filter error process $\boldsymbol{x}_{\varepsilon,k}$
${}^{S}\boldsymbol{Y}_{k}$	Gyroscope measurement (units deg/s) at iteration k
$\boldsymbol{z}_{\varepsilon,k}$	The measurement error vector at iteration k
α	Scaling constant from deg/s to radians: $$\alpha = \left(\frac{\pi\delta t}{180}\right)$$
δ_k	Geomagnetic inclination angle (deg) at iteration k
δ_k^{6DOF}	6DOF (accelerometer and magnetometer) estimate of the geomagnetic inclination angle δ at iteration k
δ_k^{+}	*A posteriori* (gyroscope) estimate of the geomagnetic inclination angle δ at iteration k
δt	Sampling interval of the Kalman filter (units s)
η	Rotation angle (degrees or radians)
$\boldsymbol{\omega}_k$	True angular velocity (deg/s)
$\boldsymbol{\omega}_k^{-}$	The *a priori* estimate of the angular velocity $\boldsymbol{\omega}_k$ (deg/s)

1.3 Software Functions

Table 1. Sensor Fusion software functions

Functions	Section
Quaternion Algebra (orientation.c)	
`void fveqconjgquq` `(struct fquaternion *pfq, float fu[], float fv[])`	3
Nine Axis Sensor Fusion (fusion.c)	

`void fInit_9DOF_GBY_KALMAN`
`(struct SV_9DOF_GBY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct MagSensor *pthisMag, struct GyroSensor *pthisGyro, struct MagCalibration *pthisMagCal);`
 	
`void fRun_9DOF_GBY_KALMAN`
`(struct SV_9DOF_GBY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct MagSensor *pthisMag, struct GyroSensor *pthisGyro, struct MagCalibration *pthisMagCal);`
	5

Six Axis Sensor Fusion (fusion.c)	
`void fInit_6DOF_GY_KALMAN`	6

Functions	Section

`(struct SV_6DOF_GY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct GyroSensor *pthisGyro);`
 	
`void fRun_6DOF_GY_KALMAN (struct SV_6DOF_GY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct GyroSensor *pthisGyro);`	

2. Estimating Acceleration and Magnetic Disturbance

2.1 Introduction

This section describes the algorithm for estimating the noise levels affecting the accelerometer estimate of the gravity vector and the magnetometer estimate of the geomagnetic vector. The estimated noise levels are then used in the Kalman filter described later to determine the relative weightings applied to the gyroscope, accelerometer and magnetometer sensor data.

The accelerometer sensor at rest has high frequency, sample to sample, measurement noise of approximately 3 m*g* and an orientation-dependent error of order 30 m*g* or so resulting from various sources including i) non-linearity in the sensor signal chain ii) imperfectly calibrated offset iii) imperfectly calibrated gain and iv) uncorrected cross-axis interference. In addition, the accelerometer will also experience large and rapidly changing physical accelerations of up to 8000 m*g* in gaming applications where the orientation can change at 2000 dps. The level of noise affecting the accelerometer estimate of the gravity vector therefore varies by approximately 60 dB.

The magnetometer sensor at rest also has high frequency, sample to sample, measurement noise of 1 or 2 μT and a low frequency orientation dependent error of a few μT resulting from an imperfect estimate of the hard and soft iron interference resulting from magnetic sources on the PCB. Magnetic disturbance, which is defined as resulting from external magnetic sources such as magnets, unlike hard and soft iron magnetic interference which are fixed in the PCB frame, can vary from near zero to 1000 μT or more when a magnet is brought close. The level of noise affecting the magnetometer estimate of the geomagnetic vector therefore also varies by about 60 dB.

At any location, the gravity vector is constant and points downwards and the geomagnetic vector is constant and points northwards and downwards (upwards) in the northern (southern) hemisphere. When measured by the accelerometer and magnetometer (calibrated for hard and soft iron effects) after rotation to an arbitrary orientation, the magnitudes of these two reference vectors measured by the two sensors is constant. The locus of the gravity vector measured in the rotated sensor frame is therefore a sphere with radius 1 *g* (in the absence of acceleration or accelerometer calibration noise) and, similarly, the locus of the geomagnetic vector measured in the rotated sensor frame is a sphere with radius equal to the geomagnetic field strength B (again, assuming no magnetic disturbance nor magnetic calibration noise).

Accelerometer and magnetometer sensor noise are zero mean and tend to move the measurement magnitudes away from the 1 *g* gravity and geomagnetic spheres respectively producing a radial error.

Accelerometer (gain, offset and cross axis) and magnetometer (hard and soft iron) calibration errors appear as orientation-dependent bulges distorting the measurement loci from spheres to ellipsoids. The larger the calibration error at any orientation, the larger the radial error deviation from the 1 *g* gravity and geomagnetic spheres.

Finally, physical acceleration and magnetic disturbances add to the accelerometer and magnetometer measurements and also move these measurements away from the gravity and geomagnetic spheres.

The instantaneous deviations of the accelerometer and magnetometer measurements away from the gravity and geomagnetic spheres can therefore be used as proxies for the vector sum of the interfering noise, from all sources, in the accelerometer and magnetometer measurements. This may be inaccurate for any specific measurement where, by chance, the interfering noise vectors cancel or move the measurement onto another point on the gravity or geomagnetic spheres but is statistically valid over a large number of measurements.

2.2 Accelerometer Sensor Model and Noise Variance

The calibrated accelerometer measurement ${}^S\boldsymbol{G}_{c,k}$ includes physical acceleration ${}^S\boldsymbol{a}_k$, gravitational ${}^S\boldsymbol{g}_k$ and noise $\boldsymbol{v}_{G,k}$ components where the noise term $\boldsymbol{v}_{G,k}$ includes all the terms discussed in the previous section.

All acceleration *sensors* are natively 'acceleration positive' in that their axes are defined such that physical acceleration in the positive direction of any axis increases the sensor output. The Android *coordinate system* is 'acceleration positive' like the accelerometer but the Aerospace / NED and Windows 8 *coordinate systems* are 'gravity positive' or ‘acceleration negative’. The sign difference between acceleration and gravity results from basic Physics in that it is impossible to distinguish the two cases of i) being at rest in a downwards pointing 1 *g* gravitational field or ii) accelerating upwards at 1 *g* in the absence of any gravitational field.

In the reference position of zero roll or pitch, the accelerometer measurement in the Aerospace / NED coordinate system is +1g since both the *z* axis and gravity are aligned and point downwards. In the Android (ENU) coordinate system, the accelerometer measurement in the same reference position is also +1 *g* since both the z axis and acceleration equivalent to gravity are aligned and point upwards. In the Windows 8 (ENU) coordinate system, the accelerometer measurement is –1 *g* since the z axis points upwards and the gravity vector points downwards.

The models for the accelerometer measurement ${}^S\boldsymbol{G}_{c,k}$ in the sensor frame for the three coordinate systems are therefore:

$${}^S\boldsymbol{G}_{c,k} = -{}^S\boldsymbol{a}_k + {}^S\boldsymbol{g}_k - \boldsymbol{v}_{G,k} \text{ (Aerospace, Windows 8)} \quad (1)$$

$${}^S\boldsymbol{G}_{c,k} = {}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k} \text{ (Android)} \quad (2)$$

The physical acceleration variance $Q_{a,k}$ is defined as:

$$Q_{a,k} = E[|{}^S\boldsymbol{a}_k|^2] \quad (3)$$

The accelerometer sensor noise variance $Q_{vG,k}$ (including high frequency sensor noise and low frequency orientation-dependent noise from calibration errors) is defined as:

$$Q_{vG,k} = E\left[\left|\boldsymbol{v}_{G,k}\right|^2\right] \quad (4)$$

For all three coordinate systems, the the sum of instantaneous acceleration variance $Q_{a,k}$ and the sensor noise variance $Q_{vG,k}$ is defined in terms of the deviation of the accelerometer measurement magnitude $\left|{}^S\boldsymbol{G}_{c,k}\right|$ from the 1 *g* sphere g as:

$$Q_{a,k} + Q_{vG,k} \approx k\left(\left|{}^{S}\boldsymbol{G}_{c,k}\right| - g\right)^2 = k\left(\left|{}^{S}\boldsymbol{a}_k - {}^{S}\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right| - g\right)^2 \tag{5}$$

where k is a constant of proportionality that can be determined from geometrical arguments.

Figure 1 shows the vector sum of the component terms in equation (5). The acceleration and sensor noise vectors ${}^{S}\boldsymbol{a}_k$ and $\boldsymbol{v}_{G,k}$ are uncorrelated and will have some distribution about the true gravity measurement in the sensor frame ${}^{S}\boldsymbol{g}_k$. The magnitude r_k of the summed vector noise terms at iteration k equals:

$$r_k = \left|{}^{S}\boldsymbol{a}_k + \boldsymbol{v}_{G,k}\right| \tag{6}$$

The angle θ_k is defined in Figure 1 such that $r_k sin\theta_k$ is the radial error of the measurement to the 1 *g* sphere.

$$r_k sin\theta_k = \left|{}^{S}\boldsymbol{a}_k - {}^{S}\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right| - g \tag{7}$$

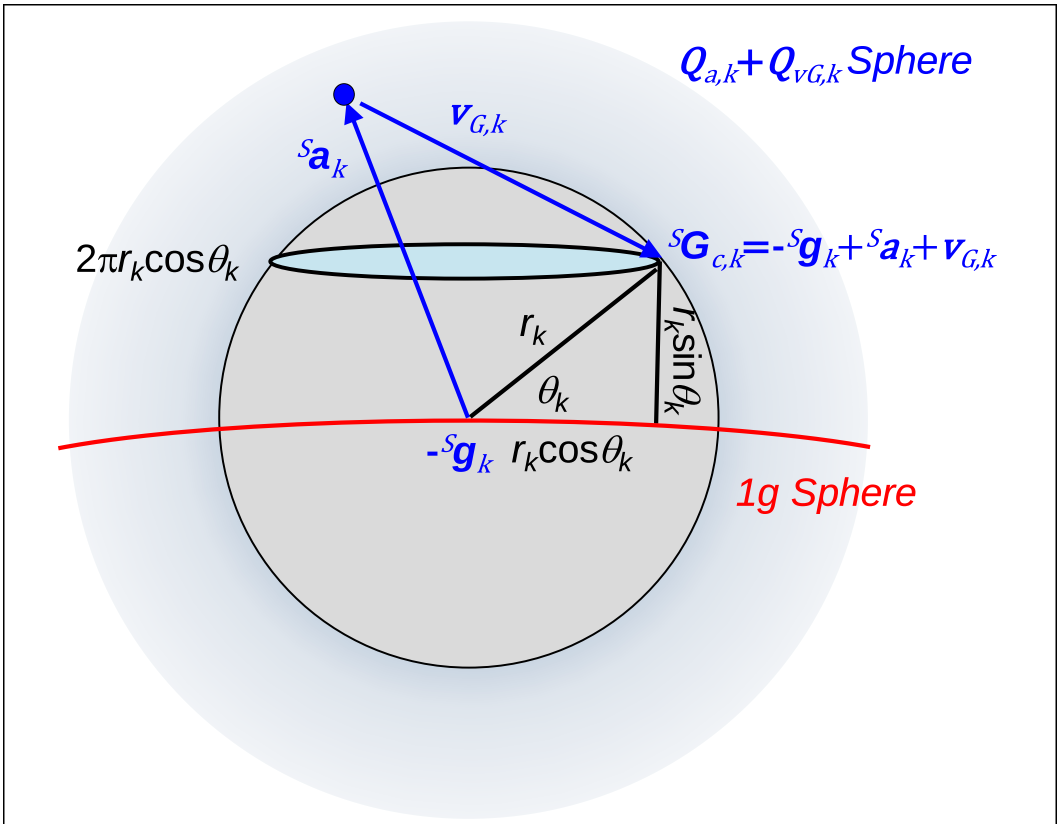

Fig 1. Geometry of acceleration and sensor noise relative to 1*g* sphere

The constant k can be calculated with the assumptions that i) the noise is spherically distributed and ii) has magnitude significantly smaller than 1 *g* giving:

$$\int_0^\infty \int_0^{\frac{\pi}{2}} p(r)r^2 2\pi r cos\theta d\theta dr = k\int_0^\infty \int_0^{\frac{\pi}{2}} p(r)r^2 sin^2\theta 2\pi r cos\theta d\theta\, dr \quad (8)$$

The function $p(r)$ defines the statistics of the falloff of the noise with amplitude but is irrelevant since the integrals separate into radial and angular terms and the radial distribution $p(r)$ cancels:

$$2\pi \int_0^\infty p(r)r^3 dr \int_0^{\frac{\pi}{2}} cos\theta d\theta = 2\pi k \int_0^\infty p(r)r^3 dr \int_0^{\frac{\pi}{2}} sin^2\theta cos\theta d\theta \quad (9)$$

$$\Rightarrow [sin\theta]_0^{\frac{\pi}{2}} = k\left[\frac{sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} \Rightarrow k = 3 \quad (10)$$

Substituting back into equation (5) gives a simple expression for the sum of the acceleration and sensor noise variances as three times the square of the difference of the magnitude of the accelerometer measurement from the 1 g sphere:

$$Q_{vG,k} + Q_{a,k} \approx 3\left(\left|{}^{S}\boldsymbol{G}_{c,k}\right| - g\right)^2 \tag{11}$$

2.3 Magnetometer Sensor Model and Noise Variance

The calibrated magnetometer measurement ${}^{S}\boldsymbol{B}_{c,k}$ is modeled as the sum of the geomagnetic component ${}^{S}\boldsymbol{m}_k$, any magnetic disturbance ${}^{S}\boldsymbol{d}_k$ and the magnetometer noise $\boldsymbol{v}_{B,k}$ which includes both sensor noise and hard and soft iron calibration errors:

$${}^{S}\boldsymbol{B}_{c,k} = {}^{S}\boldsymbol{m}_k + {}^{S}\boldsymbol{d}_k + \boldsymbol{v}_{B,k} \tag{12}$$

The magnetic calibration algorithms remove hard and soft iron magnetic distortion effects which are constant in the sensor frame leaving the calibrated measurement ${}^{S}\boldsymbol{B}_{c,k}$. The magnetic disturbance ${}^{S}\boldsymbol{d}_k$ is defined as any magnetic interference which does not rotate with the sensor frame and is not, therefore, included in the hard and soft iron calibration.

The magnetic disturbance variance $Q_{d,k}$ is defined as:

$$Q_{d,k} = E[|{}^{S}\boldsymbol{d}_k|^2] \tag{13}$$

The magnetometer sensor noise variance $Q_{vB,k}$ (including high frequency sensor noise and low frequency orientation-dependent noise from hard and soft iron calibration errors) is defined as:

$$Q_{vB,k} = E\left[\left|\boldsymbol{v}_{B,k}\right|^2\right] \tag{14}$$

With perfect magnetic calibration and in the absence of any magnetic disturbance and sensor noise, the calibrated magnetometer measurement lies on the geomagnetic sphere and has magnitude equal to the geomagnetic field strength B. In practice the measurement will not lie on the geomagnetic sphere as a consequence of i) magnetometer sensor noise ii) imperfect hard and soft iron calibration and iii) the presence of magnetic disturbance in the environment.

Using the same arguments and algebra used for the accelerometer, the sum of the magnetic disturbance and magnetometer noise variances is estimated as three times the square of the difference of the magnitude of the calibrated magnetometer measurement from the geomagnetic sphere:

$$Q_{vB,k} + Q_{d,k} \approx 3\left(\left|{}^{S}\boldsymbol{B}_{c,k}\right| - B\right)^2 \tag{15}$$

2.4 Compile Time Constants

To provide robustness against the occasional situations where the vector sum of the accelerometer and magnetometer noise components results in a measurement lying exactly on the gravity or geomagnetic sphere, the sensor fusion software applies lower bounds to the noise estimates in equations (11) and (15). These constants are listed below and defined in file fusion.h.

For the full nine degrees of freedom accelerometer, magnetometer and gyroscope Kalman filter defined in Section 5, the constants are:

```
// minimum accelerometer noise variance units g^2 computed
from 1g sphere
#define FQVG_9DOF_GBY_KALMAN 1.2E-3
// minimum magnetometer noise variance units uT^2 computed
from geomagnetic sphere
#define FQVB_9DOF_GBY_KALMAN 5E0
```

For the six degrees of freedom accelerometer and gyroscope Kalman filter defined in Section 6, the constant is:

```
// minimum accelerometer noise variance units g^2 computed
from 1g sphere
#define FQVG_6DOF_GY_KALMAN 1.2E-3
```


3. Gravity and Geomagnetic Tilt Error Quaternions

3.1 Introduction

Integrating the output from the gyroscope sensor gives the *a priori* orientation matrix and *a priori* estimates of the gravity and geomagnetic vectors in the sensor frame of reference. The gyroscope sensor is insensitive to acceleration and magnetic interference and the *a priori* estimates of the gravity and geomagnetic vectors are therefore also insensitive to acceleration and magnetic interference. But any error in the integration of the gyroscope sensor will result in a slow drift in the gyroscope's *a priori* estimates of the gravity and geomagnetic vectors requiring *a posteriori* corrections performed by the Kalman filter.

The accelerometer and magnetometer sensors can compute a six degree of freedom (or 6DOF) estimate of orientation using the approach documented in AN5021 *"Calculation of Orientation Matrices from Sensor Data*". The 6DOF orientation estimate is sensitive to acceleration and magnetic disturbance but does not suffer from long term drift. The 6DOF accelerometer and magnetometer orientation estimate is therefore complementary to the *a priori* gyroscope orientation estimate and can be used to stabilize the *a priori* orientation.

Sections 3.2 and 3.3 derive expressions for i) the gravity vector and ii) the geomagnetic vector estimates computed from the *a priori* gyroscope orientation matrix and from the 6DOF accelerometer and magnetometer orientation matrix.

Section 3.4 derives an expression for the rotation quaternion required to rotate one vector onto another vector of equal length. This expression allows the calculation of the rotation quaternions defining the tilt errors between the *a priori* gyroscope and 6DOF accelerometer and magnetometer estimates of the gravity and geomagnetic vectors. The vector components of these two tilt error quaternions form six of the nine components of the indirect Kalman filter with the gyroscope zero rate offset error providing the remaining three components.

If equation (11) indicates that the accelerometer measurement has low noise and is reliable then the *a posteriori* Kalman filter orientation correction reduces the gravity tilt error in the direction of the 6DOF gravity vector estimate. Similarly, if equation (15) indicates that the magnetometer measurement has low noise and is reliable then the *a posteriori* Kalman filter orientation correction reduces the geomagnetic tilt error in the direction of the 6DOF geomagnetic vector estimate. Since the gravity and geomagnetic vectors are not parallel, except at the geomagnetic poles, the application of the two orientation tilt corrections computed by the Kalman filter results in a stabilized *a posteriori* orientation estimate.

3.2 Gravity Vector Estimation

The gravity vector ${}^{G}\boldsymbol{g}_k$ in the global reference frame is constant and always points exactly downwards for all Kalman filter iterations k. For the Aerospace (NED) coordinate system, gravity is in the direction of the positive *z* axis and for the Android (ENU) and Windows 8 (ENU) coordinate systems, gravity is in the direction of the negative *z* axis.

The NXP convention is that the orientation matrix transforms a vector from the global frame to the sensor frame. The gravity vector ${}^{S}\boldsymbol{g}_k$ in the sensor frame can therefore be computed by multiplying ${}^{G}\boldsymbol{g}_k$ by the orientation matrix $\boldsymbol{R}_k$:

$$ {}^{S}\boldsymbol{g}_k = \boldsymbol{R}_k {}^{G}\boldsymbol{g}_k \quad (16) $$

Evaluating equation (16) using the 6DOF (accelerometer and magnetometer) orientation matrix $\widehat{\boldsymbol{R}}_k^{6DOF}$ and the *a priori* (gyroscope) orientation matrix $\widehat{\boldsymbol{R}}_k^{-}$ gives the 6DOF ${}^{S}\boldsymbol{g}_k^{6DOF}$ and *a priori* ${}^{S}\boldsymbol{g}_k^{-}$ estimates of the gravity vector in the sensor frame for the three coordinate systems as:

$$ {}^{S}\boldsymbol{g}_k^{6DOF} = \begin{pmatrix} \hat{R}_{xx,k}^{6DOF} & \hat{R}_{xy,k}^{6DOF} & \hat{R}_{xz,k}^{6DOF} \\ \hat{R}_{yx,k}^{6DOF} & \hat{R}_{yy,k}^{6DOF} & \hat{R}_{yz,k}^{6DOF} \\ \hat{R}_{zx,k}^{6DOF} & \hat{R}_{zy,k}^{6DOF} & \hat{R}_{zz,k}^{6DOF} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{R}_{xz,k}^{6DOF} \\ \hat{R}_{yz,k}^{6DOF} \\ \hat{R}_{zz,k}^{6DOF} \end{pmatrix} \; for\ Aerospace/NED \quad (17) $$

$$ {}^{S}\boldsymbol{g}_k^{-} = \begin{pmatrix} \hat{R}_{xx,k}^{-} & \hat{R}_{xy,k}^{-} & \hat{R}_{xz,k}^{-} \\ \hat{R}_{yx,k}^{-} & \hat{R}_{yy,k}^{-} & \hat{R}_{yz,k}^{-} \\ \hat{R}_{zx,k}^{-} & \hat{R}_{zy,k}^{-} & \hat{R}_{zz,k}^{-} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{R}_{xz,k}^{-} \\ \hat{R}_{yz,k}^{-} \\ \hat{R}_{zz,k}^{-} \end{pmatrix} \; for\ Aerospace/NED \quad (18) $$

$$ {}^{S}\boldsymbol{g}_k^{6DOF} = \begin{pmatrix} \hat{R}_{xx,k}^{6DOF} & \hat{R}_{xy,k}^{6DOF} & \hat{R}_{xz,k}^{6DOF} \\ \hat{R}_{yx,k}^{6DOF} & \hat{R}_{yy,k}^{6DOF} & \hat{R}_{yz,k}^{6DOF} \\ \hat{R}_{zx,k}^{6DOF} & \hat{R}_{zy,k}^{6DOF} & \hat{R}_{zz,k}^{6DOF} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} \hat{R}_{xz,k}^{6DOF} \\ \hat{R}_{yz,k}^{6DOF} \\ \hat{R}_{zz,k}^{6DOF} \end{pmatrix} \; for\ Android, Windows\ 8 \quad (19) $$

$$ {}^{S}\boldsymbol{g}_k^{-} = \begin{pmatrix} \hat{R}_{xx,k}^{-} & \hat{R}_{xy,k}^{-} & \hat{R}_{xz,k}^{-} \\ \hat{R}_{yx,k}^{-} & \hat{R}_{yy,k}^{-} & \hat{R}_{yz,k}^{-} \\ \hat{R}_{zx,k}^{-} & \hat{R}_{zy,k}^{-} & \hat{R}_{zz,k}^{-} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} \hat{R}_{xz,k}^{-} \\ \hat{R}_{yz,k}^{-} \\ \hat{R}_{zz,k}^{-} \end{pmatrix} \; for\ Android, Windows\ 8 \quad (20) $$

3.3 Geomagnetic Vector Estimation

The geomagnetic vector ${}^{G}\boldsymbol{m}_k$ in the global reference frame is constant and points northwards and downwards from horizontal by the inclination angle δ_k. For the Aerospace (NED) coordinate system, northwards is in the direction of the positive *x* axis and for the Android (ENU) and Windows 8 (ENU) coordinate systems, northwards is in the direction of the positive *y* axis.

The geomagnetic vector ${}^{S}\boldsymbol{m}_k$ in the sensor frame can be computed by multiplying ${}^{G}\boldsymbol{m}_k$ by the orientation matrix $\boldsymbol{R}_k$:

$$ {}^{S}\boldsymbol{m}_k = \boldsymbol{R}_k {}^{G}\boldsymbol{m}_k \quad (21) $$

Evaluating equation (21) using the 6DOF (accelerometer and magnetometer) orientation matrix $\widehat{\boldsymbol{R}}_k^{6DOF}$ and the *a priori* (gyroscope) orientation matrix $\widehat{\boldsymbol{R}}_k^{-}$ gives the 6DOF ${}^{S}\boldsymbol{m}_k^{6DOF}$ and *a priori* ${}^{S}\boldsymbol{m}_k^{-}$ estimates of the geomagnetic vector in the sensor frame for the three coordinate systems as:

$$\frac{{}^S\boldsymbol{m}_k^{6DOF}}{B} = \widehat{\boldsymbol{R}}_k^{6DOF}\begin{pmatrix} cos\delta_k^{6DOF} \\ 0 \\ sin\delta_k^{6DOF} \end{pmatrix} = \begin{pmatrix} \hat{R}_{xx,k}^{6DOF} cos\delta_k^{6DOF} + \hat{R}_{xz,k}^{6DOF} sin\delta_k^{6DOF} \\ \hat{R}_{yx,k}^{6DOF} cos\delta_k^{6DOF} + \hat{R}_{yz,k}^{6DOF} sin\delta_k^{6DOF} \\ \hat{R}_{zx,k}^{6DOF} cos\delta_k^{6DOF} + \hat{R}_{zz,k}^{6DOF} sin\delta_k^{6DOF} \end{pmatrix} for\ Aerospace/NED \quad (22)$$

$$\frac{{}^S\boldsymbol{m}_k^{-}}{B} = \widehat{\boldsymbol{R}}_k^{-}\begin{pmatrix} cos\delta_k^{6DOF} \\ 0 \\ sin\delta_k^{6DOF} \end{pmatrix} = \begin{pmatrix} \hat{R}_{xx,k}^{-} cos\delta_{k-1}^{+} + \hat{R}_{xz,k}^{-} sin\delta_{k-1}^{+} \\ \hat{R}_{yx,k}^{-} cos\delta_{k-1}^{+} + \hat{R}_{yz,k}^{-} sin\delta_{k-1}^{+} \\ \hat{R}_{zx,k}^{-} cos\delta_{k-1}^{+} + \hat{R}_{zz,k}^{-} sin\delta_{k-1}^{+} \end{pmatrix} for\ Aerospace/NED \quad (23)$$

$$\frac{{}^S\boldsymbol{m}_k^{6DOF}}{B} = \widehat{\boldsymbol{R}}_k^{6DOF}\begin{pmatrix} 0 \\ cos\delta_k^{6DOF} \\ -sin\delta_k^{6DOF} \end{pmatrix} = \begin{pmatrix} \hat{R}_{xy,k}^{6DOF} cos\delta_k^{6DOF} - \hat{R}_{xz,k}^{6DOF} sin\delta_k^{6DOF} \\ \hat{R}_{yy,k}^{6DOF} cos\delta_k^{6DOF} - \hat{R}_{yz,k}^{6DOF} sin\delta_k^{6DOF} \\ \hat{R}_{zy,k}^{6DOF} cos\delta_k^{6DOF} - \hat{R}_{zz,k}^{6DOF} sin\delta_k^{6DOF} \end{pmatrix} for\ Android, Windows\ 8 \quad (24)$$

$$\frac{{}^S\boldsymbol{m}_k^{-}}{B} = \widehat{\boldsymbol{R}}_k^{-}\begin{pmatrix} 0 \\ cos\delta_k^{6DOF} \\ -sin\delta_k^{6DOF} \end{pmatrix} = \begin{pmatrix} \hat{R}_{xy,k}^{-} cos\delta_{k-1}^{+} - \hat{R}_{xz,k}^{-} sin\delta_{k-1}^{+} \\ \hat{R}_{yy,k}^{-} cos\delta_{k-1}^{+} - \hat{R}_{yz,k}^{-} sin\delta_{k-1}^{+} \\ \hat{R}_{zy,k}^{-} cos\delta_{k-1}^{+} - \hat{R}_{zz,k}^{-} sin\delta_{k-1}^{+} \end{pmatrix} for\ Android, Windows\ 8 \quad (25)$$

δ_k^{6DOF} is the 6DOF estimate of the inclination angle at iteration k and δ_{k-1}^{+} is the *a posteriori* estimate of the inclination angle from the previous iteration $k-1$. The previous iteration's *a posterori* estimate δ_{k-1}^{+} is used simply because the *a posteriori* inclination angle δ_k^{+} for the current iteration is not available until the Kalman filter has executed.

AN5021 "*Calculation of Orientation Matrices from Sensor Data*" equations (77), (88) and (99) derive how the 6DOF inclination angle estimate δ_k^{6DOF} can be computed from the scalar product of the accelerometer ${}^S\boldsymbol{G}_{c,k}$ and magnetometer ${}^S\boldsymbol{B}_{c,k}$ measurements as:

$$sin(\delta_k^{6DOF}) = \frac{{}^S\boldsymbol{G}_{c,k}.\ {}^S\boldsymbol{B}_{c,k}}{|{}^S\boldsymbol{G}_{c,k}||{}^S\boldsymbol{B}_{c,k}|}\ for\ Aerospace\ (NED), Windows\ 8 \quad (26)$$

$$sin(\delta_k^{6DOF}) = \frac{-\,{}^S\boldsymbol{G}_{c,k}.\ {}^S\boldsymbol{B}_{c,k}}{|{}^S\boldsymbol{G}_{c,k}||{}^S\boldsymbol{B}_{c,k}|}\ for\ Android \quad (27)$$

The *a posteriori* inclination angle estimate δ_k^{+} is computed from the scalar product of the normalized *a posteriori* gravity ${}^S\boldsymbol{g}_k^{+}$ and geomagnetic vector ${}^S\boldsymbol{m}_k^{+}$ estimates:

$$sin(\delta_k^{+}) = \frac{{}^S\boldsymbol{g}_k^{+}.\ {}^S\boldsymbol{m}_k^{+}}{|{}^S\boldsymbol{g}_k^{+}||{}^S\boldsymbol{m}_k^{+}|}\ for\ Aerospace\ (NED), Android, Windows\ 8 \quad (28)$$

The angle sine rather than cosine in equations (26) to (28) results from the definition of the inclination angle as the dip of the magnetic field below horizontal and not the angle with the vertical gravity vector.

3.4 Rotation Quaternion Between Two Vectors

Sections 3.2 and 3.3 derived expressions for the 6DOF (accelerometer and magnetometer) and *a priori* (gyroscope) estimates of the gravity and geomagnetic vectors measured in the sensor frame. This section derives the rotation quaternions that relate these two sets of vector measurements and specifically the quaternions that rotate the 6DOF estimates of the gravity and geomagnetic vectors onto the *a priori* estimates.

The rotation quaternion q required to rotate a general vector $\boldsymbol{r}$ onto vector $\boldsymbol{s}$ satisfies:

$$\boldsymbol{s} = q^{*}\boldsymbol{r}q \quad \mathbf{(29)}$$

For there to be a solution to equation (29), the magnitude of both vectors must be equal $|\boldsymbol{s}| = |\boldsymbol{r}|$ since a vector's magnitude is unchanged under rotation.

The angle η between the two vectors can be determined from the scalar product $\boldsymbol{r}.\boldsymbol{s}$:

$$cos\eta = 2cos^{2}\left(\frac{\eta}{2}\right) - 1 = \frac{\boldsymbol{r}.\boldsymbol{s}}{|\boldsymbol{r}||\boldsymbol{s}|} \quad \mathbf{(30)}$$

Rearranging gives the solution for the scalar component q_0 of the rotation quaternion q:

$$q_0 = cos\left(\frac{\eta}{2}\right) = \sqrt{\frac{1}{2} + \frac{\boldsymbol{r}.\boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|}} = \sqrt{\frac{|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|}} \quad \mathbf{(31)}$$

The rotation axis $\hat{\boldsymbol{n}}$ is calculated from the vector product $\boldsymbol{r} \times \boldsymbol{s}$:

$$\hat{\boldsymbol{n}}sin\eta = 2\hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right)cos\left(\frac{\eta}{2}\right) = \frac{-\boldsymbol{r} \times \boldsymbol{s}}{|\boldsymbol{r}||\boldsymbol{s}|} \quad \mathbf{(32)}$$

The minus sign in equation (32) derives from the requirement to calculate the rotation axis $\hat{\boldsymbol{n}}$ for a coordinate system rotation by angle η rather than the rotation axis required to rotate the vector $\boldsymbol{r}$ onto $\boldsymbol{s}$ in a fixed coordinate system.

Substitution gives the vector component $\boldsymbol{q} = \{q_1, q_2, q_3\} = \hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right)$ of the rotation quaternion q:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right) = \frac{-\boldsymbol{r} \times \boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|\sqrt{\frac{|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|}}} \quad \mathbf{(33)}$$

The required rotation quaternion q is then:

$$q = cos\left(\frac{\eta}{2}\right) + \hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right) = \sqrt{\frac{|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|}} - \frac{\boldsymbol{r} \times \boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|\sqrt{\frac{|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s}}{2|\boldsymbol{r}||\boldsymbol{s}|}}} = \frac{|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s} - \boldsymbol{r} \times \boldsymbol{s}}{\sqrt{2|\boldsymbol{r}||\boldsymbol{s}|(|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s})}} \quad \mathbf{(34)}$$

The solution for the scalar component of the rotation quaternion q_0 in equation (31) is always defined except for the nonsensical case of zero magnitude vectors $|\boldsymbol{r}| = |\boldsymbol{s}| = 0$.

The solution for the vector component $\boldsymbol{q}$ of the rotation quaternion in equation (33) is undefined when $|\boldsymbol{r}||\boldsymbol{s}| + \boldsymbol{r}.\boldsymbol{s} = 0$ which occurs when the two vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are anti-parallel. In this case the numerator is also zero since $\boldsymbol{r} \times \boldsymbol{s} = 0$. The rotation angle between the two vectors is 180º but there are an infinite number of possible rotation axes orthogonal to $\boldsymbol{r}$ and $\boldsymbol{s}$.

By inspection, one solution (of the infinite number available) for the vector component $\boldsymbol{q}$ for the 180º rotation case valid for all cases except $r_x = r_y = r_z$ is:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\sqrt{\left(r_y - r_z\right)^2 + (r_z - r_x)^2 + \left(r_x - r_y\right)^2}} \begin{pmatrix} r_y - r_z \\ r_z - r_x \\ r_x - r_y \end{pmatrix} \quad \textbf{(35)}$$

The vector quaternion $\boldsymbol{q}$ in equation (35) has magnitude $|\boldsymbol{q}| = 1$ corresponding to 180º rotation and is obviously orthogonal to the vector $\boldsymbol{r}$ (and therefore also orthogonal to $\boldsymbol{s} = -\boldsymbol{r}$) since:

$$\boldsymbol{q}.\boldsymbol{r} = \frac{1}{\sqrt{\left(r_y - r_z\right)^2 + (r_z - r_x)^2 + \left(r_x - r_y\right)^2}} \begin{pmatrix} r_y - r_z \\ r_z - r_x \\ r_x - r_y \end{pmatrix} . \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = 0 \quad \textbf{(36)}$$

For the special case $r_x = r_y = r_z$ and 180º rotation angle, a solution is:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \textbf{(37)}$$

since $|\boldsymbol{q}| = 1$ and $\boldsymbol{q}.\boldsymbol{r} = 0$.

For the case when the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ have unit magnitude, equation (34) can be simplified to:

$$q = cos\left(\frac{\eta}{2}\right) + \hat{\boldsymbol{n}} sin\left(\frac{\eta}{2}\right) = \frac{1 + \boldsymbol{r}.\boldsymbol{s} - \boldsymbol{r} \times \boldsymbol{s}}{\sqrt{2(1 + \boldsymbol{r}.\boldsymbol{s})}} = \frac{1}{\sqrt{2}}\left(\sqrt{1 + \boldsymbol{r}.\boldsymbol{s}} - \frac{\boldsymbol{r} \times \boldsymbol{s}}{\sqrt{(1 + \boldsymbol{r}.\boldsymbol{s})}}\right) \quad \textbf{(38)}$$
$$for\ |\boldsymbol{r}| = |\boldsymbol{s}| = 1$$

Equation (38) is implemented in function `fveqconjgquq` in file *orientation.c*.

3.5 Gravity and Geomagnetic Tilt Error Quaternions

Sections 3.2, 3.3 and 3.4 can now be combined to give the rotation quaternions relating the 6DOF and *a priori* estimates of the gravity and geomagnetic vectors measured in the sensor frame. These quaternions are termed *tilt error* quaternions since they are the measurement *error* vectors input to the Kalman filter which define the *tilt* angles between the two estimates of the gravity and geomagnetic vectors.

The normalized form of equation (34) in equation (38) can be used since the gravity and normalized geomagnetic vectors defined in equations (17) to (20) and equations (22) to (25) have unit magnitude.

Substituting equations (17) to (20) into equation (38) gives an expression for the gravity tilt error quaternion $q_{zg\varepsilon,k}$ as:

$$q_{zg\varepsilon,k} = \frac{1}{\sqrt{2}}\left(\sqrt{1 + {}^{S}\boldsymbol{g}_k^{6DOF} . {}^{S}\boldsymbol{g}_k^{-}} - \frac{{}^{S}\boldsymbol{g}_k^{6DOF} \times {}^{S}\boldsymbol{g}_k^{-}}{\sqrt{\left(1 + {}^{S}\boldsymbol{g}_k^{6DOF} . {}^{S}\boldsymbol{g}_k^{-}\right)}}\right) \tag{39}$$

with vector component $\boldsymbol{q}_{zg\varepsilon,k}$:

$$\boldsymbol{q}_{zg\varepsilon,k} = \frac{- {}^{S}\boldsymbol{g}_k^{6DOF} \times {}^{S}\boldsymbol{g}_k^{-}}{\sqrt{\left(2 + 2\, {}^{S}\boldsymbol{g}_k^{6DOF} . {}^{S}\boldsymbol{g}_k^{-}\right)}} \tag{40}$$

Substituting equations (22) to (25) into equation (35) gives an expression for the geomagnetic tilt error quaternion $q_{zm\varepsilon,k}$ as:

$$q_{zm\varepsilon,k} = \frac{1}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{{}^{S}\boldsymbol{m}_k^{6DOF}}{B}\right).\left(\frac{{}^{S}\boldsymbol{m}_k^{-}}{B}\right)} - \frac{\left(\frac{{}^{S}\boldsymbol{m}_k^{6DOF}}{B}\right) \times \left(\frac{{}^{S}\boldsymbol{m}_k^{-}}{B}\right)}{\sqrt{\left\{1 + \left(\frac{{}^{S}\boldsymbol{m}_k^{6DOF}}{B}\right).\left(\frac{{}^{S}\boldsymbol{m}_k^{-}}{B}\right)\right\}}}\right] \tag{41}$$

with vector component $\boldsymbol{q}_{zm\varepsilon,k}$:

$$\boldsymbol{q}_{zm\varepsilon,k} = \frac{-\left(\frac{{}^{S}\boldsymbol{m}_k^{6DOF}}{B}\right) \times \left(\frac{{}^{S}\boldsymbol{m}_k^{-}}{B}\right)}{\sqrt{2 + 2\left(\frac{{}^{S}\boldsymbol{m}_k^{6DOF}}{B}\right).\left(\frac{{}^{S}\boldsymbol{m}_k^{-}}{B}\right)}} \tag{42}$$

4. Gyroscope Sensor Model

4.1 Zero Rate Offset Model

The model of the gyroscope sensor measurement ${}^{S}\boldsymbol{Y}_k$ (with units deg/s) is:

$$ {}^{S}\boldsymbol{Y}_k = \boldsymbol{\omega}_k + \boldsymbol{b}_k + \boldsymbol{v}_{Y,k} \quad (43) $$

where $\boldsymbol{\omega}_k$ is the true angular velocity in deg/s and $\boldsymbol{b}_k$ the gyroscope offset vector (deg/s). $\boldsymbol{v}_{Y,k}$ is the additive gyroscope noise vector (deg/s) with covariance matrix $\boldsymbol{Q}_{vY}$ assumed to be constant over time, uncorrelated between axes (diagonal) and to have the same value $\frac{Q_{vY}}{3}$ in each axis:

$$ \boldsymbol{Q}_{vY} = E\left[\boldsymbol{v}_{Y,k}\left(\boldsymbol{v}_{Y,k}\right)^T\right] = \left(\frac{Q_{vY}}{3}\right)\boldsymbol{I} \quad (44) $$

The gyroscope offset $\boldsymbol{b}_k$ vector (units deg/s) is modeled as the random walk:

$$ \boldsymbol{b}_k = \boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k} \quad (45) $$

where $\boldsymbol{w}_{b,k}$ is a zero mean white Gaussian noise vector with units of deg/s with covariance Q_{wb} assumed to be constant over time, uncorrelated between axes (diagonal) and to have the same value $\frac{Q_{wb}}{3}$ in each axis:

$$ \boldsymbol{Q}_{wb} = E\left[\boldsymbol{w}_{b,k}\left(\boldsymbol{w}_{b,k}\right)^T\right] = \left(\frac{Q_{wb}}{3}\right)\boldsymbol{I} \quad (46) $$

The *a priori* estimate of the gyroscope offset is simply the *a posteriori* estimate from the previous sample since $\boldsymbol{w}_{b,k}$ is zero mean and white:

$$ \hat{\boldsymbol{b}}_k^- = \hat{\boldsymbol{b}}_{k-1}^+ \quad (47) $$

Simple algebra gives $\hat{\boldsymbol{b}}_{\varepsilon,k}^-$ as a function of $\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+$:

$$ \hat{\boldsymbol{b}}_{\varepsilon,k}^- = \hat{\boldsymbol{b}}_k^- - \boldsymbol{b}_k = \hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_k = \hat{\boldsymbol{b}}_{k-1}^+ - \left(\boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k}\right) = \left(\hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_{k-1}\right) - \boldsymbol{w}_{b,k} \quad (48) $$

$$ \Rightarrow \hat{\boldsymbol{b}}_{\varepsilon,k}^- = \hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ - \boldsymbol{w}_{b,k} \quad (49) $$

4.2 Angular Velocity Model

The *a priori estimate* $\hat{\boldsymbol{\omega}}_k^-$ of the true angular velocity $\boldsymbol{\omega}_k$ is computed from the gyroscope reading in the sensor frame ${}^S\boldsymbol{Y}_k$ by subtracting off the current *a priori* zero rate gyroscope offset $\hat{\boldsymbol{b}}_k^-$ which equals the *a posteriori* offset estimate $\hat{\boldsymbol{b}}_{k-1}^+$ from the previous iteration:

$$\hat{\boldsymbol{\omega}}_k^- = \left({}^S\boldsymbol{Y}_k - \hat{\boldsymbol{b}}_k^-\right) = \left({}^S\boldsymbol{Y}_k - \hat{\boldsymbol{b}}_{k-1}^+\right) \tag{50}$$

Substituting for ${}^S\boldsymbol{Y}_k$ gives the relationship between the error components:

$$\hat{\boldsymbol{\omega}}_k^- = \boldsymbol{\omega}_k + \boldsymbol{b}_k + \boldsymbol{v}_{Y,k} - \hat{\boldsymbol{b}}_{k-1}^+ \tag{51}$$

By definition:

$$\hat{\boldsymbol{\omega}}_k^- = \boldsymbol{\omega}_k + \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- \tag{52}$$

$$\Rightarrow \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- = \boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} - \hat{\boldsymbol{b}}_{k-1}^+ \tag{53}$$

By definition:

$$\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ = \hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_{k-1} \tag{54}$$

$$\Rightarrow \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- = -\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} = -\hat{\boldsymbol{b}}_{\varepsilon,k}^- + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} \tag{55}$$

Equation (55) states that the error in the *a priori* estimate of angular velocity comprises three terms:

i) The error in the *a priori* estimate $\hat{\boldsymbol{b}}_{\varepsilon,k}^-$ of the gyroscope zero rate sensor. The minus sign results from the fact that an over-estimate of the gyroscope zero rate gyroscope leads to an under-estimate of the angular velocity.

ii) The noise $\boldsymbol{w}_{b,k}$ in the gyroscope zero rate offset drift.

iii) The additive gyroscope sensor noise $\boldsymbol{v}_{Y,k}$.

The sensor noise term $\boldsymbol{v}_{Y,k}$ can only be separated from the offset drift term $\boldsymbol{w}_{b,k}$ by observing the gyroscope over a period long enough for the drift to be measurable. On a sample by sample basis, the gyroscope offset drift term $\boldsymbol{w}_{b,k}$ is indistinguishable from the gyroscope noise term $\boldsymbol{v}_{Y,k}$.

4.3 Compile Time Constants

The value of Q_{vY} is set for the 6DOF and 9DOF Kalman filter algorithms in the compile time constants `FQVY_9DOF_GBY_KALMAN` and `FQVY_6DOF_GY_KALMAN` defined in file *fusion.h*. Increasing the value of Q_{vY} gives a lower weighting to the gyroscope orientation estimate which results in more rapid convergence to the gravity and geomagnetic vector estimate from the accelerometer and magnetometer but increased sensitivity to acceleration and magnetic disturbance noise.

The value of Q_{wb} is set for the 6DOF and 9DOF Kalman filter algorithms in the compile time constants `FQWB_9DOF_GBY_KALMAN` and `FQWB_6DOF_GY_KALMAN` defined in file *fusion.h*. Increasing the value of Q_{wb} allows more rapid tracking of changes in the zero rate offset, including the initial estimation at power on, but has the drawback of increased sensitivity to acceleration and magnetic disturbance noise.

5. Accelerometer, Magnetometer and Gyroscope Sensor Fusion Kalman Filter

5.1 Introduction

Section 2 derived expressions for the interfering noise levels from sensor noise and from external sources such as acceleration and magnetic disturbance. Section 3 defined the two tilt error quaternions which form the measurement error vector in the complementary Kalman filter. Section 4 defined the gyroscope and angular velocity models.

This section combines all these results to derive the Kalman filter equations for the sensor fusion of accelerometer, magnetometer and gyroscope data. It is also commonly referred to as 9 degree of freedom or 9DOF sensor fusion since each of the three sensors has three axes and provides 3 degrees of freedom to the filter.

5.2 Direct Kalman Filter Process Model

The system is modeled with the seven element state vector $\boldsymbol{x}_k$ comprising i) the orientation quaternion q_k and ii) the zero rate gyroscope offset $\boldsymbol{b}_k$ at iteration k:

$$\boldsymbol{x}_k = \begin{pmatrix} q_k \\ \boldsymbol{b}_k \end{pmatrix} \tag{56}$$

The orientation quaternion component q_k of the state vector evolves over the Kalman filter time period δt from iteration $k-1$ to k through the system angular velocity $\boldsymbol{\omega}_k$ as:

$$q_k = q_{k-1}\Delta q(\boldsymbol{\omega}_k \delta t) \tag{57}$$

$\Delta q(\boldsymbol{\omega}_k \delta t)$ is the incremental rotation quaternion encoding rotation by angle $|\boldsymbol{\omega}_k|\delta t$ about normalized rotation axis $\hat{\boldsymbol{n}} = \left(\frac{\boldsymbol{\omega}_k}{|\boldsymbol{\omega}_k|}\right)$:

$$\Delta q = \left\{ cos\left(\frac{|\boldsymbol{\omega}_k|\delta t}{2}\right), \left(\frac{\boldsymbol{\omega}_k}{|\boldsymbol{\omega}_k|}\right) sin\left(\frac{|\boldsymbol{\omega}_k|\delta t}{2}\right) \right\} \tag{58}$$

The gyroscope offset component $\boldsymbol{b}_k$ of the state vector evolves as the random walk defined in equation (45).

5.3 A Priori Estimation of the Direct Process Model

The *a priori* estimate $\hat{q}_k^-$ of the orientation quaternion at iteration k is computed using equation (57) by rotating the previous *a posteriori* orientation estimate $\hat{q}_{k-1}^+$ by the incremental rotation vector $\hat{\boldsymbol{\omega}}_k^- \delta t$ during the Kalman filter interval δt:

$$\hat{q}_k^- = \hat{q}_{k-1}^+ \Delta q(\hat{\boldsymbol{\omega}}_k^- \delta t) \tag{59}$$

where the *a priori* estimate of the angular velocity $\hat{\boldsymbol{\omega}}_k^-$ is defined in equation (50).

The *a priori* estimate of the gyroscope zero rate offset is given by equation (47) and simply equals the *a posteriori* estimate from the previous iteration.

5.4 Indirect Kalman Filter Process Model

Instead of estimating the process $\boldsymbol{x}_k$ directly, it is more convenient to use an *indirect* Kalman filter whose state vector is the 9x1 error vector $\boldsymbol{x}_{\varepsilon,k}$ with components:

$$\boldsymbol{x}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{q}_{m\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} \tag{60}$$

The 3x1 vector $\boldsymbol{q}_{g\varepsilon,k}$ is the vector component of the quaternion $q_{g\varepsilon,k}$ which models the orientation error in tilt angle relative to the true gravity vector ${}^S\boldsymbol{g}_k$. The term ‘gravity tilt error angle’ has its normal meaning of a tilt error relative to the downwards pointing gravity vector. It can be determined using accelerometer measurements alone.

The 3x1 vector $\boldsymbol{q}_{m\varepsilon,k}$ is the vector component of the quaternion $q_{m\varepsilon,k}$ which models the orientation error in tilt angle relative to the true geomagnetic vector ${}^S\boldsymbol{m}_k$. The term ‘geomagnetic tilt error angle’ is analogous to the gravity tilt angle but is now defined as the tilt relative to the northwards and downwards (upwards) pointing geomagnetic vector in the northern (southern) hemisphere. It can be determined using magnetometer measurements alone.

The 3x1 vector $\boldsymbol{b}_{\varepsilon,k}$ (deg/s) models the error in the estimate of the zero rate gyroscope offset.

The simultaneous correction by the Kalman filter of both the gravity and geomagnetic tilt error angles corrects the estimated orientation towards the true orientation. In the presence of high levels of magnetic disturbance from a magnet, only the gravity tilt error will be corrected resulting in an orientation estimate stable in roll and pitch but susceptible to drift in compass heading over a sufficiently long period. Similarly in the presence of high levels of acceleration disturbance from shaking, only the geomagnetic tilt error will be corrected leading to an orientation estimate susceptible to drift in roll and pitch over a sufficiently long period.

5.5 A Posteriori Correction of the Direct Process Model

The indirect Kalman filter computes *a posteriori* estimate $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ of the error state vector $\boldsymbol{x}_{\varepsilon,k}$ defined in equation (60):

$$\boldsymbol{x}_{\varepsilon,k}^{+} = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^{+} \\ \hat{\boldsymbol{q}}_{m\varepsilon,k}^{+} \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^{+} \end{pmatrix} \tag{61}$$

This section documents how the *a posteriori* error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ is used to compute the *a posteriori* state vector $\boldsymbol{x}_k^{+}$ defined as:

$$\boldsymbol{x}_k^+ = \begin{pmatrix} \hat{q}_k^+ \\ \widehat{\boldsymbol{b}}_k^+ \end{pmatrix} \tag{62}$$

The *a posteriori* gravity tilt error quaternion $\hat{q}_{g\varepsilon,k}^+$ is used to correct the *a priori* estimate of the gravity vector in the sensor frame ${}^S\boldsymbol{g}_k^-$ defined in equations (18) and (20) as:

$${}^S\boldsymbol{g}_k^+ = \hat{q}_{g\varepsilon,k}^+ \, {}^S\boldsymbol{g}_k^- \left(\hat{q}_{g\varepsilon,k}^+\right)^* \tag{63}$$

The *a posteriori* geomagnetic tilt error quaternion $\hat{q}_{m\varepsilon,k}^+$ is used to correct the *a priori* estimate of the geomagnetic vector in the sensor frame ${}^S\boldsymbol{m}_k^-$ defined in equations (23) and (25) as:

$${}^S\boldsymbol{m}_k^+ = \hat{q}_{m\varepsilon,k}^+ \, {}^S\boldsymbol{m}_k^- \left(\hat{q}_{m\varepsilon,k}^+\right)^* \tag{64}$$

Equations (63) and (64) are vector rotations performed by pre- and post-multiplication by a rotation quaternion and its conjugate. The conjugated rotation quaternion appears on the right hand side and the non-conjugated quaternion on the left to ensure that the tilt error is *removed* from the current *a priori* estimate rather than added.

The *a posteriori* estimate of the gyroscope offset vector $\widehat{\boldsymbol{b}}_k^+$ is simply the *a priori* estimate minus the *a posteriori* error estimate $\widehat{\boldsymbol{b}}_{\varepsilon,k}^+$:

$$\widehat{\boldsymbol{b}}_k^+ = \widehat{\boldsymbol{b}}_k^- - \widehat{\boldsymbol{b}}_{\varepsilon,k}^+ = \widehat{\boldsymbol{b}}_{k-1}^+ - \widehat{\boldsymbol{b}}_{\varepsilon,k}^+ \tag{65}$$

The final step is the straightforward calculation of the *a posteriori* orientation quaternion $\hat{q}_k^+$ from the *a posteriori* gravity vector ${}^S\boldsymbol{g}_k^+$ and geomagnetic vector ${}^S\boldsymbol{m}_k^+$ estimates computed in equations (63) and (64). This is done by first calculating the *a posteriori* orientation matrix $\widehat{\boldsymbol{R}}_k^+$ using the vector product algorithm documented in Section 6 of AN5021 "*Calculation of Orientation Matrices from Sensor Data*" and then calculating the orientation quaternion $\hat{q}_k^+$ from the orientation matrix $\widehat{\boldsymbol{R}}_k^+$.

5.6 Kalman Filter Measurement Error Model

The vector components of the tilt error quaternions $\boldsymbol{q}_{zg\varepsilon,k}$ and $\boldsymbol{q}_{zm\varepsilon,k}$ are defined in equations (40) and (42) and are measured between the 6DOF (accelerometer and magnetometer) and *a priori* (gyroscope) estimates of the gravity and geomagnetic vectors. These two vector quaternions form the components of the 6x1 indirect Kalman filter measurement error vector $\boldsymbol{z}_{\varepsilon,k}$:

$$\boldsymbol{z}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{zg\varepsilon,k} \\ \boldsymbol{q}_{zm\varepsilon,k} \end{pmatrix} \tag{66}$$

The measurement error vector $\boldsymbol{z}_{\varepsilon,k}$ is modeled as being related to the error process vector $\boldsymbol{x}_{\varepsilon,k}$ through the 6x9 measurement matrix $\boldsymbol{C}_k$ plus measurement noise $\boldsymbol{v}_k$:

$$\boldsymbol{z}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{zg\varepsilon,k} \\ \boldsymbol{q}_{zm\varepsilon,k} \end{pmatrix} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k = \boldsymbol{C}_k \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{q}_{m\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} + \begin{pmatrix} \boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k} \end{pmatrix} \quad (67)$$

where the 6x1 measurement noise vector $\boldsymbol{v}_k$ is decomposed into 3x1 gravity $\boldsymbol{v}_{qzg,k}$ and 3x1 geomagnetic $\boldsymbol{v}_{qzm,k}$ measurement noise vectors:

$$\boldsymbol{v}_k = \begin{pmatrix} \boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k} \end{pmatrix} \quad (68)$$

The measured gravity tilt error quaternion $q_{zg\varepsilon,k}$ measures the difference between the *a priori* (gyroscope) and 6DOF (accelerometer and magnetometer) orientation estimates. It is therefore equal to the product of the true gravity tilt error quaternion $q_{g\varepsilon,k}$ and i) the quaternion error $q\left(\widehat{\boldsymbol{\omega}}_{\varepsilon,k}^{-}\delta t\right)$ resulting from an erroneous estimate of the *a priori* angular velocity and ii) the quaternion error $q(\boldsymbol{v}_k)$ introduced by noise in 6DOF orientation estimate:

$$q_{zg\varepsilon,k} = q_{g\varepsilon,k}\, q\left(\widehat{\boldsymbol{\omega}}_{\varepsilon,k}^{-}\delta t\right) q(\boldsymbol{v}_k) = q_{g\varepsilon,k}\, q\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\delta t\right) q\left(\boldsymbol{w}_{b,k}\delta t\right) q\left(\boldsymbol{v}_{Y,k}\delta t\right) q\left(\boldsymbol{v}_{qzg,k}\right) \quad (69)$$

With the assumption that the errors are small, the scalar components of the quaternions are near unity and equation (69) can be written as:

$$\left\{1, \boldsymbol{q}_{zg\varepsilon,k}\right\} \approx \left\{1, \boldsymbol{q}_{g\varepsilon,k}\right\}\left\{1, \boldsymbol{q}\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\delta t\right)\right\}\left\{1, \boldsymbol{q}\left(\boldsymbol{w}_{b,k}\delta t\right)\right\}\left\{1, \boldsymbol{q}\left(\boldsymbol{v}_{Y,k}\delta t\right)\right\}\left\{1, \boldsymbol{q}\left(\boldsymbol{v}_{qzg,k}\right)\right\} \quad (70)$$

Separating out the quaternion vector components of equation (70) gives:

$$\boldsymbol{q}_{zg\varepsilon,k} \approx \boldsymbol{q}_{g\varepsilon,k} + \left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right) + \boldsymbol{v}_{qzg,k} \quad (71)$$

The factor $\left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)$ converts from the native units of deg/s in the gyro offset error, random walk and noise vectors to the sine of half the subtended angle (equal to half the subtended angle in radians) used in the quaternion vector.

Similar arguments for the measured geomagnetic tilt error vector quaternion $\boldsymbol{q}_{zm\varepsilon,k}$ give:

$$\boldsymbol{q}_{zm\varepsilon,k} \approx \boldsymbol{q}_{m\varepsilon,k} + \left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right) + \boldsymbol{v}_{qmg,k} \quad (72)$$

With the definition of the constant α as:

$$\alpha = \left(\frac{\pi\delta t}{180}\right) \quad (73)$$

the 6x9 measurement matrix $\boldsymbol{C}_k$ can be written as:

$$\boldsymbol{C}_k = \begin{pmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_3 & \left(\frac{-\alpha}{2}\right)\boldsymbol{I}_3 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \left(\frac{-\alpha}{2}\right)\boldsymbol{I}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) \\ 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) \end{pmatrix} \quad (74)$$

5.7 Indirect Kalman Filter Update Equations

This section references the standard Kalman filter equations derived in AN5018 “Basic Kalman Filter Theory” but applied to the error process in the indirect Kalman filter described in this section.

The error process $\boldsymbol{x}_{\varepsilon,k}$ defined in equation (60) is modeled as evolving in conventional Kalman filter style using the linear model:

$$\boldsymbol{x}_{\varepsilon,k} = \boldsymbol{A}_k \boldsymbol{x}_{\varepsilon,k-1} + \boldsymbol{w}_k \quad (75)$$

The matrix $\boldsymbol{A}_k$ is the Kalman filter state matrix and $\boldsymbol{w}_k$ is an additive unpredictable (white) noise component.

Equation (A) in AN5018 defines the *a priori* estimate of the error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ in terms of the previous iteration’s *a posteriori* estimate $\hat{\boldsymbol{x}}_{\varepsilon,k-1}^{+}$ as:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} = \boldsymbol{A}_k \hat{\boldsymbol{x}}_{\varepsilon,k-1}^{+} \quad (76)$$

Equations (63) to (65) for iteration $k-1$ use the *a posteriori* error vector $\boldsymbol{x}_{\varepsilon,k-1}^{+}$ to correct the *a posteriori* state vector $\boldsymbol{x}_{k-1}^{+}$ with the result that the *a priori* error vector estimate $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ for the next iteration k is zero:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} = 0 \Rightarrow \boldsymbol{A}_k = 0 \quad (77)$$

Equation (D) in AN5018 defines the *a posteriori* estimate of the error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ in terms of the measurement matrix $\boldsymbol{C}_k$, Kalman gain matrix $\boldsymbol{K}_k$ and measurement vector $\boldsymbol{z}_{\varepsilon,k}$ as:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{+} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}_k)\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} + \boldsymbol{K}_k \boldsymbol{z}_{\varepsilon,k} \quad (78)$$

Substituting equation (77) gives:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{+} = \boldsymbol{K}_k \boldsymbol{z}_{\varepsilon,k} \tag{79}$$

Equation (B1) in AN5018 defines the evolution of the *a priori* covariance matrix $\boldsymbol{P}_k^{-}$ as:

$$\boldsymbol{P}_k^{-} = \boldsymbol{A}_k \boldsymbol{P}_{k-1}^{+} {\boldsymbol{A}_k}^T + \boldsymbol{Q}_{w,k} \tag{80}$$

where $\boldsymbol{Q}_{w,k}$ is the covariance matrix of the process noise vector $\boldsymbol{w}_k$:

$$\boldsymbol{Q}_{w,k} = E[\boldsymbol{w}_k {\boldsymbol{w}_k}^T] \tag{81}$$

Substituting $\boldsymbol{A}_k = \boldsymbol{0}$ from equation (77) gives:

$$\boldsymbol{P}_k^{-} = \boldsymbol{Q}_{w,k} \tag{82}$$

Equation (C) of AN5018 defines the Kalman gain matrix $\boldsymbol{K}_k$ as:

$$\boldsymbol{K}_k = \boldsymbol{P}_k^{-} {\boldsymbol{C}_k}^T \left(\boldsymbol{C}_k \boldsymbol{P}_k^{-} {\boldsymbol{C}_k}^T + \boldsymbol{Q}_{v,k}\right)^{-1} \tag{83}$$

where $\boldsymbol{Q}_{v,k}$ is the covariance matrix of the measurement noise vector $\boldsymbol{v}_k$:

$$\boldsymbol{Q}_{v,k} = E[\boldsymbol{v}_k {\boldsymbol{v}_k}^T] \tag{84}$$

Substituting equation (82) gives the expression for the Kalman filter gain in the indirect filter as:

$$\boldsymbol{K}_k = \boldsymbol{Q}_{w,k} {\boldsymbol{C}_k}^T \left(\boldsymbol{C}_k \boldsymbol{Q}_{w,k} {\boldsymbol{C}_k}^T + \boldsymbol{Q}_{v,k}\right)^{-1} \tag{85}$$

The indirect Kalman filter is now completely defined once expressions are derived for the two noise covariance matrices $\boldsymbol{Q}_{w,k}$ and $\boldsymbol{Q}_{v,k}$. But before deriving these two matrices in the next section, it's useful to understand their role in the operation of the indirect Kalman filter.

The covariance matrix $\boldsymbol{Q}_{w,k}$ models the errors in the *a priori* extrapolation of the state vector using the gyroscope sensor and the covariance matrix $\boldsymbol{Q}_{v,k}$ models the errors in the measurement of the state vector using the accelerometer and magnetometer sensors.

In the limiting case of high measurement noise covariance $\boldsymbol{Q}_{v,k}$, the Kalman filter should use the *a priori* gyroscope sensor extrapolation and apply zero *a posteriori* correction from the measurement vector $\boldsymbol{z}_{\varepsilon,k}$. Evaluating equations (85) and (78) in this limit gives the expected result:

$$\boldsymbol{K}_k \approx \boldsymbol{Q}_{w,k}\boldsymbol{C}_k{}^T\boldsymbol{Q}_{v,k}{}^{-1} = \boldsymbol{0} \tag{86}$$

$$\Rightarrow \widehat{\boldsymbol{x}}_{\varepsilon,k}^{+} = \widehat{\boldsymbol{x}}_{\varepsilon,k}^{-} \tag{87}$$

In the limiting case of low measurement noise covariance $\boldsymbol{Q}_{v,k}$, the Kalman filter should ignore the gyroscope and use the orientation estimate measured from the accelerometer and magnetometer. Evaluating equations (85) and (78) in this limit gives the expected result:

$$\boldsymbol{C}_k\boldsymbol{K}_k\boldsymbol{C}_k \approx \left(\boldsymbol{C}_k\boldsymbol{Q}_{w,k}\boldsymbol{C}_k{}^T\right)\left(\boldsymbol{C}_k\boldsymbol{Q}_{w,k}\boldsymbol{C}_k{}^T\right)^{-1}\boldsymbol{C}_k = \boldsymbol{C}_k \Rightarrow \boldsymbol{K}_k\boldsymbol{C}_k = \boldsymbol{I}_9 \tag{88}$$

$$\Rightarrow \widehat{\boldsymbol{x}}_{\varepsilon,k}^{+} = \boldsymbol{K}_k\boldsymbol{z}_{\varepsilon,k} \tag{89}$$

5.8 Process Error Covariance Matrix

The 9x9 error process noise covariance matrix $\boldsymbol{Q}_{w,k}$ measures the error covariance in the *a priori* linear prediction of the error state vector $\boldsymbol{x}_{\varepsilon,k}$ defined in equation (75) from one iteration to the next:

$$\boldsymbol{Q}_{w,k} = E[\boldsymbol{w}_k\boldsymbol{w}_k{}^T] = \begin{pmatrix} E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \\ E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \\ E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \end{pmatrix} \tag{90}$$

The covariance $E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right]$ evaluates to:

$$E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] = E\left[\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)^T\right] = \boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + \left(\frac{Q_{wb}}{3}\right)\boldsymbol{I}_3 \tag{91}$$

$\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+}$ is approximated with the computed *a posteriori* values at iteration $k-1$ ignoring off-diagonal terms

$$\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} = E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\right)^T\right] \approx \begin{pmatrix} \left(\hat{b}_{x\varepsilon,k-1}^{+}\right)^2 & 0 & 0 \\ 0 & \left(\hat{b}_{y\varepsilon,k-1}^{+}\right)^2 & 0 \\ 0 & 0 & \left(\hat{b}_{z\varepsilon,k-1}^{+}\right)^2 \end{pmatrix} \tag{92}$$

The *a priori* gravity tilt error covariance $E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right]$ evaluates to:

$$E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] = E\left[\left\{\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}\left\{\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}^T\right] \tag{93}$$

$$= \boldsymbol{Q}_{q_{g\varepsilon}q_{g\varepsilon},k-1}^{+} + \left(\frac{\alpha}{2}\right)^2\left\{\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + \left(\frac{Q_{vY}}{3}\right)\boldsymbol{I}_3 + \left(\frac{Q_{wb}}{3}\right)\boldsymbol{I}_3\right\} \tag{94}$$

where $\boldsymbol{Q}^{+}_{q_{g\varepsilon}q_{g\varepsilon},k-1}$ is approximated by the *a posteriori* values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{q_{g\varepsilon}q_{g\varepsilon},k-1} = E\left[\hat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1}\left(\hat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1}\right)^T\right] \approx \begin{pmatrix} \left(\hat{q}^{+}_{gx\varepsilon,k-1}\right)^2 & 0 & 0 \\ 0 & \left(\hat{q}^{+}_{gy\varepsilon,k-1}\right)^2 & 0 \\ 0 & 0 & \left(\hat{q}^{+}_{gz\varepsilon,k-1}\right)^2 \end{pmatrix} \quad \textbf{(95)}$$

Similarly, the geomagnetic tilt error covariance $E\left[\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\left(\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\right)^T\right]$ can be written as:

$$E\left[\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\left(\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\right)^T\right] = E\left[\hat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1} + \left(\frac{\alpha}{2}\right)\left(-\hat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\left\{\hat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1} + \left(\frac{\alpha}{2}\right)\left(-\hat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}^T\right] \quad \textbf{(96)}$$

$$= \boldsymbol{Q}^{+}_{q_{m\varepsilon}q_{m\varepsilon},k-1} + \left(\frac{\alpha}{2}\right)^2\left\{\boldsymbol{Q}^{+}_{b_\varepsilon b_\varepsilon,k-1} + \left(\frac{Q_{vY}}{3}\right)\boldsymbol{I}_3 + \left(\frac{Q_{wb}}{3}\right)\right\} \quad \textbf{(97)}$$

where $\boldsymbol{Q}^{+}_{q_{m\varepsilon}q_{m\varepsilon},k-1}$ is approximated by the *a posteriori* values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{q_{m\varepsilon}q_{m\varepsilon},k-1} = E\left[\hat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1}\left(\hat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1}\right)^T\right] \approx \begin{pmatrix} \left(\hat{q}^{+}_{mx\varepsilon,k-1}\right)^2 & 0 & 0 \\ 0 & \left(\hat{q}^{+}_{my\varepsilon,k-1}\right)^2 & 0 \\ 0 & 0 & \left(\hat{q}^{+}_{mz\varepsilon,k-1}\right)^2 \end{pmatrix} \quad \textbf{(98)}$$

The cross covariance of gravity tilt and geomagnetic tilt errors is assumed uncorrelated and the covariance $E\left[\hat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\right)^T\right]$ is set to zero.

The covariance $E\left[\hat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\hat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^T\right]$ evaluates to:

$$E\left[\hat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\hat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^T\right] = E\left[\left\{\hat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1} + \left(\frac{\alpha}{2}\right)\left(-\hat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}\left(\hat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} - \boldsymbol{w}_{b,k}\right)^T\right] \quad \textbf{(99)}$$

$$= \boldsymbol{Q}^{+}_{g_\varepsilon b_\varepsilon,k-1} - \left(\frac{\alpha}{2}\right)\left\{\boldsymbol{Q}^{+}_{b_\varepsilon b_\varepsilon,k-1} + \left(\frac{Q_{wb}}{3}\right)\boldsymbol{I}_3\right\} \quad \textbf{(100)}$$

where $\boldsymbol{Q}^{+}_{g_\varepsilon b_\varepsilon,k-1}$ is approximated with the *a posteriori* values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{g_\varepsilon b_\varepsilon,k-1} = E\left[\hat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1}\left(\hat{\boldsymbol{b}}^{+}_{\varepsilon,k-1}\right)^T\right] \approx \begin{pmatrix} \hat{q}^{+}_{gx\varepsilon,k-1}\hat{b}^{+}_{x\varepsilon,k-1} & 0 & 0 \\ 0 & \hat{q}^{+}_{gy\varepsilon,k-1}\hat{b}^{+}_{y\varepsilon,k-1} & 0 \\ 0 & 0 & \hat{q}^{+}_{gz\varepsilon,k-1}\hat{b}^{+}_{z\varepsilon,k-1} \end{pmatrix} \quad \textbf{(101)}$$

The covariance $E\left[\hat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\left(\hat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^T\right]$ evaluates to:

$$E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^{T}\right] = E\left[\left\{\widehat{\boldsymbol{q}}_{m\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)^{T}\right] \quad (102)$$

$$= \boldsymbol{Q}_{m_\varepsilon b_\varepsilon,k-1}^{+} - \left(\frac{\alpha}{2}\right)\left\{\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + \left(\frac{Q_{wb}}{3}\right)\boldsymbol{I}_3\right\} \quad (103)$$

where $\boldsymbol{Q}_{m_\varepsilon b_\varepsilon,k-1}^{+}$ is approximated with the *a posteriori* values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}_{m_\varepsilon b_\varepsilon,k-1}^{+} = E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k-1}^{+}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\right)^{T}\right] \approx \begin{pmatrix} \hat{q}_{mx\varepsilon,k-1}^{+}\hat{b}_{x\varepsilon,k-1}^{+} & 0 & 0 \\ 0 & \hat{q}_{my\varepsilon,k-1}^{+}\hat{b}_{y\varepsilon,k-1}^{+} & 0 \\ 0 & 0 & \hat{q}_{mz\varepsilon,k-1}^{+}\hat{b}_{z\varepsilon,k-1}^{+} \end{pmatrix} \quad (104)$$

5.9 Measurement Error Covariance Matrix

The 6x6 covariance matrix of the measurement noise vector $\boldsymbol{v}_k$ is defined using equation (68) as:

$$\boldsymbol{Q}_{v,k} = E[\boldsymbol{v}_k \boldsymbol{v}_k{}^T] = E\left[\begin{pmatrix}\boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k}\end{pmatrix}^T\right] = \begin{pmatrix} E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] & E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzm,k}\right)^T\right] \\ E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] & E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzm,k}\right)^T\right] \end{pmatrix} \quad (105)$$

The measurement quaternion vector $\boldsymbol{q}_{zg\varepsilon,k}$ is proportional to the sine of *half* the rotation angle between the *a priori* and 6DOF measurements of the gravity vector. Its noise term $\boldsymbol{v}_{qzg,k}$ therefore includes i) the accelerometer sensor noise plus acceleration noise and ii) the gyroscope sensor and zero rate offset noise. It is unaffected by magnetometer noise and magnetic disturbance.

Similarly, the measurement quaternion vector $\boldsymbol{q}_{zm\varepsilon,k}$ is the sine of *half* the rotation angle between the *a priori* and 6DOF measurements of the geomagnetic vector and its noise term $\boldsymbol{v}_{qzm,k}$ therefore includes i) the magnetometer sensor noise plus magnetic disturbance noise and ii) the gyroscope sensor and zero rate offset noise. It is unaffected by accelerometer noise and acceleration.

With a small angle approximation and, remembering that the native units of the vector quaternion are radians and the native units of the gyroscope are deg/s, then the terms in $\boldsymbol{Q}_{v,k}$ are:

$$E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] = \left(\frac{1}{4}\right)\left\{\left(\frac{Q_{vG,k}}{3}\right) + \left(\frac{Q_{a,k}}{3}\right)\right\}\boldsymbol{I}_3 + \left(\frac{\alpha^2}{4}\right)\left\{\left(\frac{Q_{vY,k}}{3}\right) + \left(\frac{Q_{wb,k}}{3}\right)\right\}\boldsymbol{I}_3 \quad (106)$$

$$= \left(\frac{1}{12}\right)\left\{\left(Q_{vG,k} + Q_{a,k}\right) + \alpha^2\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \quad (107)$$

$$E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzm,k}\right)^T\right] = \left(\frac{1}{4B^2}\right)\left\{\left(\frac{Q_{vB,k}}{3}\right) + \left(\frac{Q_{d,k}}{3}\right)\right\}\boldsymbol{I}_3 + \left(\frac{\alpha^2}{4}\right)\left\{\left(\frac{Q_{vY,k}}{3}\right) + \left(\frac{Q_{wb,k}}{3}\right)\right\}\boldsymbol{I}_3 \quad (108)$$

$$= \left(\frac{1}{12}\right)\left\{\frac{\left(Q_{vB,k} + Q_{d,k}\right)}{B^2} + \alpha^2\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \quad \textbf{(109)}$$

The noise covariances $Q_{vG,k} + Q_{a,k}$ and $Q_{vB,k} + Q_{d,k}$ are defined in equations (11) and (15) in terms of the measurement deviations from the gravity and geomagnetic spheres.

The cross-correlation measurement noise terms are assumed to be uncorrelated:

$$E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzm,k}\right)^T\right] = \left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] \quad \textbf{(110)}$$

5.10 Compile Time Constants

These compile time constants are implemented with #define in file *fusion.h.*

The constants `FQVY_9DOF_GBY_KALMAN` and `FQWB_9DOF_GBY_KALMAN` define the covariances Q_{vY} and Q_{wb}.

The constants `FMIN_9DOF_GBY_BPL` and `FMAX_9DOF_GBY_BPL` limit the permissible range of the gyroscope zero rate offset $\boldsymbol{b}_k$. The default range is -7 deg/s to +7 deg/s. The main purpose is to prevent the gyroscope zero rate offset being initialized to a nonsensical value if the sensors are being rotated rather than held stationary when the sensor fusion is initialized.

6. Accelerometer and Gyroscope Sensor Fusion Kalman Filter

6.1 Introduction

This section derives the Kalman filter equations for the sensor fusion of accelerometer and gyroscope data. It is also commonly referred to as a 6 degree of freedom or 6DOF sensor fusion model since each of the two sensors has three axes providing three degrees of freedom. This Kalman filter is a simplified version of the 9DOF filter described in Section 5.

6.2 Direct Kalman Filter Process Model

This is identical to the description is Section 5.2.

6.3 A Priori Estimation of the Direct Process Model

This is identical to the description in Section 5.3 and produces the *a priori* orientation quaternion $\hat{q}_k^-$ by incremental rotation by the gyroscope angular velocity vector.

6.4 Indirect Kalman Filter Process Model

This is a simplified version of the model in Section 5.4 in that only gravity vector error quaternion is included in the error state vector.

$$\boldsymbol{x}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} \tag{111}$$

6.5 A Posteriori Correction of the Direct Process Model

This is a simplified version of the description in Section 5.5 in that only the gravity vector quaternion correction is included in the *a posteriori* correction vector.

$$\boldsymbol{x}_{\varepsilon,k}^{+} = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^{+} \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^{+} \end{pmatrix} \tag{112}$$

The *a posteriori* gravity tilt error quaternion $\hat{q}_{g\varepsilon,k}^{+}$ corrects the *a priori* orientation quaternion $\hat{q}_k^-$ directly to give the *a posteriori* orientation estimate $\hat{q}_k^+$:

$$\hat{q}_k^+ = \hat{q}_{g\varepsilon,k}^{+}\hat{q}_k^-\left(\hat{q}_{g\varepsilon,k}^{+}\right)^* \tag{113}$$

6.6 Kalman Filter Measurement Error Model

This is a simplified version of the measurement vector described in section 5.6 in that only the gravity vector tilt error quaternion is present in the measurement error vector $\boldsymbol{z}_{\varepsilon,k}$:

$$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{q}_{zg\varepsilon,k} \tag{114}$$

The measurement model is now:

$$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{q}_{zg\varepsilon,k} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k = \boldsymbol{C}_k \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} + \boldsymbol{v}_{qzg,k} \tag{115}$$

where the 3x6 measurement matrix $\boldsymbol{C}_k$ equals:

$$1\ g\boldsymbol{C}_k = \begin{pmatrix} \boldsymbol{I}_3 & \left(\frac{-\alpha}{2}\right)\boldsymbol{I}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) \end{pmatrix} \tag{116}$$

6.7 Indirect Kalman Filter Update Equations

This section is identical to Section 5.7 but using the matrix and vector definitions in Section 6.

6.8 Process Error Covariance Matrix

The 6x6 error process noise covariance matrix $\boldsymbol{Q}_{w,k}$ is a simplified version of that defined in section 5.8.

$$\boldsymbol{Q}_{w,k} = E[\boldsymbol{w}_k \boldsymbol{w}_k{}^T] = \begin{pmatrix} E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \\ E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \end{pmatrix} \tag{117}$$

The terms in the matrix have the same values as those defined in Section 5.8.

6.9 Measurement Error Covariance Matrix

The 3x3 covariance matrix of the measurement noise vector $\boldsymbol{v}_k$ is a simplified version of that defined in Section 5.9 and contains only the gravity vector term.

$$\boldsymbol{Q}_{v,k} = E[\boldsymbol{v}_k \boldsymbol{v}_k{}^T] = E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] \tag{118}$$

$$= \left(\frac{1}{12}\right)\left\{\left(Q_{vG,k} + Q_{a,k}\right) + \alpha^2\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \tag{119}$$

The noise covariance $Q_{vG,k} + Q_{a,k}$ is defined in the same as in Section 5.9 in terms of the measurement deviation from the gravity sphere using equation (11).

6.10 Compile Time Constants

The compile time constants below are implemented with #define in file *fusion.h.*

The constants `FQVY_6DOF_GY_KALMAN` and `FQWB_6DOF_GY_KALMAN` define the covariances Q_{vY} and Q_{wb}.

The constants `FMIN_6DOF_GBY_BPL` and `FMAX_6DOF_GBY_BPL` have the function function and values as their equivalent for the 9DOF fusion algorithm.

7. References

- NXP Application Note (AN5018) *Basic Kalman Filter Theory*
- NXP Application Note (AN5021) *Calculation of Orientation Matrices from Sensor Data*

8. Legal information

8.1 Definitions

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9. List of tables

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